Properties of the eigenfunctions of the SFS operator with $\alpha = 0$

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Outline

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Introduction

- I have been working on a functional linear operator called the Stretch-Fold-Shear (SFS) operator

- Model of magnetic field growth in kinematic dynamo theory [Bayly and Childress, 1988] and [Gilbert, 2002]

- The existence of an eigenvalue λ of the SFS operator of modulus greater than one ensures dynamo growth

The SFS Operator

- $\mathcal{F} = \{c(x)$ complex-valued function of a real variable $x \in [-1, 1]\}$
- SFS operator $S : \mathcal{F} \to \mathcal{F}$
- $S(c)(x) = e^{i\alpha\frac{(x-1)}{2}} c\left(\frac{x-1}{2}\right) - e^{i\alpha\frac{(1-x)}{2}} c\left(\frac{1-x}{2}\right)$
- $\alpha \geq 0$ (real parameter) is the shear parameter
- For $\alpha = 0$, $S(c)(x) = c\left(\frac{x-1}{2}\right) - c\left(\frac{1-x}{2}\right)$

Eigenvalue-Eigenfunction Pairs

k	Degree	λ	$c(x)$
0	1	1	$x-1$
1	3	$1/4$	x^3-3x^2-x+3
2	5	$1/16$	$x^5-5\,x^4-\frac{10\,x^3}{3}+30\,x^2+\frac{7\,x}{3}-25$
3	7	$1/64$	$x^7-7\,x^6-7\,x^5+105\,x^4+\frac{49\,x^3}{3}-525\,x^2-\frac{31\,x}{3}+427$
k	$2k+1$	$1/4^k$	Polynomial of degree $2k+1$

Table: The eigenvalue-eigenfunction pairs for $\alpha=0$.

All in a Bag!


Why carry the eigenfunctions separately?
Why not put them all in a bag and just carry the bag only!

Generating Function-the Bag!

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- A generating function $G(x)$ is a formal power series

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

whose coefficients give the sequence $\{a_0, a_1, ...\}$

- Transformation of sequences into functions

- Constitution of a generating function for the polynomial eigenfunctions of the SFS operator with $\alpha = 0$

- Using the generating function to find general properties of the eigenfunctions

Generating Function for the Eigenfunctions

- Perron-Frobenius (PF) operator [Vepstas, 2014] for the binary shift given by

$$\mathcal{L}f(x) = \frac{1}{2}\left[f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right)\right]$$

- Similar to the SFS operator with $\alpha = 0$ given by

$$Sc(x) = c\left(\frac{x-1}{2}\right) - c\left(\frac{1-x}{2}\right)$$

Generating Function for the Eigenfunctions

- Bernoulli polynomials $B_n(x)$ [Wikipedia, 2021] and [Wolfram MathWorld, 2021] are the eigenfunctions of the PF operator with eigenvalues $\lambda_n = 2^{-n}$

- The first Bernoulli polynomials are

$$B_0 = 1$$

$$B_1 = x - \frac{1}{2}$$

$$B_2 = x^2 - x + \frac{1}{6}x$$

$$B_3 = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

Generating Function for the Eigenfunctions

- The generating function of the Bernoulli polynomials:

$$G_B(x,t) = \frac{e^{xt}}{g(t)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \text{ where } g(t) = \frac{e^t - 1}{t}.$$

- We assume a generating function for SFS eigenfunctions of the following form:

$$G(x,t) = \frac{e^{(x-1)t} - e^{(1-x)t}}{g(t)}.$$

Generating Function for the Eigenfunctions

- We need to find the function $g(t)$ so that,

$$G(x,t) = \sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!} \text{ and } SG(x,t) = G(x,bt)$$

$$\text{i.e. } S\left(\sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!}\right) = \sum_{n=0}^{\infty} c_n(x)b^n\frac{t^n}{n!}$$

where $c_n(x)$ is an eigenfunction of degree n with eigenvalue b^n.

Generating Function for the Eigenfunctions

The generating function for the eigenfunctions is given by

$$G(x,t) = \frac{2t[e^{(x-1)t} - e^{(1-x)t}]}{e^{2t} - e^{-2t}}$$

This generating function gives all the eigenfunctions of S for $\alpha = 0$ as the coefficients of $\frac{t^n}{n!}$.

The Eigenfunctions and Bernoulli Polynomials

- c_n can be expressed in terms of B_n as follows:

$$c_n(x) = \begin{cases} 2^{2n} B_n(\frac{x+1}{4}), & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \tag{1}$$

- For even n, we don't have any eigenfunction c_n of the SFS operator.

- For odd n, B_n can also be expressed in terms of c_n as follows:

$$B_n(x) = 2^{-2n} c_n(4x - 1). \tag{2}$$

Properties of the Eigenfunctions

- Evaluating eigenfunctions at various points to see whether there are common properties

- Verifying the properties using generating functions

- Finding properties using the relationship with Bernoulli polynomials

Properties of the Eigenfunctions

Difference

1. $c_n(x+2) - c_n(x) = -2$(even part of $c_n(x)$)

2. $c_n(x+4) - c_n(x) = 4n(x+1)^{n-1}$

Multiplication

3. $c_n(mx) = m^{n-1} \sum_{k=0}^{m-1} c_n(x + \frac{4k+1}{m} - 1)$ where $m \in \mathbb{N}, m \neq 0$

Properties of the Eigenfunctions

Symmetry

4. $c_n(b - x) = -c_n(a + x)$ where $a + b = 2$ and $a, b \in \mathbb{R}$

5. $c_n(b-x) = -c_n\,(a+x)+4n(\frac{a-b}{2}+x)^{n-1}$ where $a+b = 6$ and $a, b \in \mathbb{R}$

Properties of the Eigenfunctions

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Explicit formula

6. $c_m(x) = 2^{2m} \sum_{n=0}^{m} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{x+1}{4} + k \right)^m$

Properties of the Eigenfunctions

Sums of the powers of consecutive integers

7. $\sum_{k=0}^{m-1} k^{n-1} = \frac{1}{2^{2n}n} c_n(4m-1)$ where $m \in \mathbb{N}, m \neq 0$

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Fourier series

8. $c_n(x) = -2^{n+1} n! \sum_{k=1}^{\infty} \frac{\cos\left(\frac{k\pi x}{2} + \frac{(k-n)\pi}{2}\right)}{(k\pi)^n}$ where $-1 < x < 3$

Sum identity

9. $\sum_{k=0}^{n}(-1)^{k+n}\binom{n}{k}c_n(4k-1) = 2^{2n}n!$

Properties of the Eigenfunctions

Derivative

10. $\frac{d^2}{dx^2} c_n(x) = n(n-1)c_{n-2}(x)$

Properties of the Eigenfunctions

Single integral

11. $\int_{a-x}^{b+x} c_n(u)\,du = 0$ where $a + b = 2$ and $a, b \in \mathbb{R}$

12. $\int_{3-x}^{3+x} c_n(u)\,du = 4x^n$

13. $\int_{x-1}^{x+3} c_n(u)\,du = 4x^n$

Properties of the Eigenfunctions

Double integral

14. $\int_{a-x}^{x} \int_{a-z}^{z} c_n(u)\, du\, dz = 0$

15. $\int_{-x}^{1+x} \int_{z}^{1+z} c_n(u)\, du\, dz = 0$

16. $\int_{x}^{1-x} \int_{z}^{1+z} c_n(u)\, du\, dz = 0$

17. $\int_{-x}^{1+x} \int_{z}^{1-z} c_n(u)\, du\, dz = 0$

Properties of the Eigenfunctions

Properties of the Eigenfunctions

I have randomly chosen just one of the properties to verify here:

$c_n(b-x) = -c_n(a+x)$ where $a+b=2$ and $a, b \in \mathbb{R}$

Properties of the Eigenfunctions

$c_n(b-x) = -c_n(a+x)$ where $a+b=2$ and $a, b \in \mathbb{R}$

Remember that, the generating function for the eigenfunctions is

$$G(x,t) = \sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!} = \frac{2t[e^{(x-1)t} - e^{(1-x)t}]}{e^{2t} - e^{-2t}}$$

Now, $$\sum_{n=0}^{\infty} -c_n(a+x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} -c_n(2-b+x)\frac{t^n}{n!}$$

$$= -G(2-b+x,t)$$

$$= -\frac{2t}{e^{2t} - e^{-2t}}\left[e^{(2-b+x-1)t} - e^{(1-2+b-x)t}\right]$$

Properties of the Eigenfunctions

$c_n(b-x) = -c_n(a+x)$ where $a+b=2$ and $a, b \in \mathbb{R}$

$$\sum_{n=0}^{\infty} -c_n(a+x)\frac{t^n}{n!} = \frac{2t}{e^{2t}-e^{-2t}}\left[e^{(b-x-1)t} - e^{(1-b+x)t}\right]$$

$$= G(b-x,t)$$

$$= \sum_{n=0}^{\infty} c_n(b-x)\frac{t^n}{n!}$$

Equating the coefficients of like powers of t^n from both sides we get,

$$\boxed{c_n(b-x) = -c_n(a+x) \text{ where } a+b=2.}$$

Conclusion

- Obtaining a way to generate the polynomial eigenfunctions of the SFS operator (for $\alpha = 0$)

- Finding a relationship of these polynomials with the Bernoulli polynomials

- Discovering some properties for these polynomials

Future Work

- Investigate whether there are any analogous generating methods for $\alpha > 0$
- Computer-assisted proofs for $\alpha > 0$

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The Bernoulli Operator

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Thank You